JOURNAL OF APPROXIMATION THEORY **48**, 354–360 (1986)

On Generalized Hausdorff Matrices*

DAVID BORWEIN, F. PETER CASS, AND JOHN E. SAYRE

Department of Mathematics, University of Western Ontario, London, Ontario, N6A 5B9, Canada

Communicated by R. Bojanic

Received February 23, 1984; revised March 20, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

INTRODUCTION

Given a matrix $A = \{a_{n,k}\}$ $(n, k = 0, 1, 2, \ldots)$ and a sequence $\{s_k\}$, the notation $s_n \to s(A)$ means that $\sum_{k=0}^{\infty} a_{n,k} s_k$ converges for $n = 0, 1, 2, \ldots$ and tends to s as $n \to \infty$. The matrix A is said to be regular if $s_n \to s(A)$ whenever $s_n \to s$. Necessary and sufficient conditions for A to be regular are

$$\sup_n \sum_{k=0}^{\infty} |a_{n,k}| < \infty;$$

$$\lim_{n \to \infty} a_{n,k} = 0, \qquad k = 0, 1, 2, \ldots;$$

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} = 1.$$

Suppose throughout that $\{\lambda_n\}$ is a sequence with

$$\lambda_0 \geqslant 0 \text{ and } \lambda_n > 0 \text{ for } n > 0. \tag{1}$$

Let Ω be a simply connected region that contains every positive λ_n, and suppose that, for $n = 0, 1, 2, \ldots$, Γ_n is a positively sensed Jordan contour lying in Ω and enclosing every $\lambda_k \in \Omega$ with $0 \leqslant k \leqslant n$. Suppose that f is holomorphic in Ω and that $f(\lambda_0)$ is defined even when $\lambda_0 \notin \Omega$. Define

$$\begin{aligned} \lambda_{n,k} &= -\lambda_{k+1} \cdots \lambda_n \frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(z)\, dz}{(\lambda_k - z) \cdots (\lambda_n - z)} + \delta_k \qquad &&\text{for } 0 \leqslant k \leqslant n, \\ &= 0 &&\text{for } k > n, \end{aligned} \tag{2}$$

* This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

0021-9045/86 $3.00

where $\delta_k = f(\lambda_0)$ if $k=0$ and $\lambda_0 \notin \Omega$, and $\delta_k = 0$ otherwise. Here and elsewhere the convention that products like $\lambda_{k+1} \cdots \lambda_n = 1$ when $k=n$ is observed. In many applications f is a Mellin transform

$$f(z) = \int_0^1 t^z d\alpha(t) \tag{3}$$

where $\alpha \in BV$, the space of functions of bounded variation on $[0, 1]$. In this case the region Ω in which f is holomorphic contains $\{z\colon \operatorname{Re}(z) > 0\}$; if $0 = \lambda_0 \notin \Omega$ and, with this f in (2), the order of integration is changed, then the value of $\lambda_{n,k}$ is unaffected by allowing Γ_n also to enclose λ_0 and taking $\delta_0 = 0$.

Matrices whose entries are given by (2) are called generalized Hausdorff matrices. The most familiar examples are those for which $f(z)$ is given by (3). If $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n$, $\lambda_n \to \infty$ and $\sum_{n=1}^{\infty} (1/\lambda_n) = \infty$, then (2) yields the matrices considered by Hausdorff in [3]; further, if $\lambda_n = n$, they are the matrices discussed by Hausdorff in [2] (see also Hardy [4]). The latter include the familiar Cesàro, Hölder, and Euler matrices.

For $0 < t \leqslant 1$, let $\lambda_{n,k}(t)$ denote the value of $\lambda_{n,k}$ obtained from (2) by taking $f(z) = t^z$, and let $\lambda_{n,k}(0) = \lambda_{n,k}(0+)$. Note that, from the theory of residues, $\lambda_{n,k}(t)$, for $t > 0$, is a linear combination of the functions $t^{\lambda_s} \log^r(t)$, $s = 0, 1, 2, \ldots$, $r = 0, 1, 2, \ldots$, the coefficient of t^{λ_0} being 1 when $\lambda_0 = 0$. Hence, since $\lambda_s > 0$ for $s \geqslant 1$,

$$\begin{aligned} \lambda_{n,k}(0) &= 1 \qquad \text{if } k=0 \text{ and } \lambda_0 = 0, \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{4}$$

(cf. [1, p. 947]).

Let

$$D_0 = (1 + \lambda_0)\, d_0 = 1, \tag{5}$$

$$D_n = \left(1 + \frac{1}{\lambda_1}\right)\left(1 + \frac{1}{\lambda_2}\right) \cdots \left(1 + \frac{1}{\lambda_n}\right) = (1 + \lambda_n)\, d_n \quad \text{for } n \geqslant 1. \tag{6}$$

Then, for $n \geqslant 0$,

$$D_n = \lambda_{n+1}\, d_{n+1} = 1 - d_0 + \sum_{k=0}^{n} d_k. \tag{7}$$

It is known that if all the λ_n's are different, then

$$\int_0^1 \lambda_{n,k}(t)\, dt = d_k / D_n \qquad \text{for} \quad 0 \leqslant k \leqslant n. \tag{8}$$

See [3, p. 294]. A simple continuity argument applied to (2), with $f(z)=t^z$, shows that (8) remains valid when different λ_n's are allowed to coalesce. The generalized Hausdorff matrix $M_d=\{a_{n,k}\}$ with $a_{n,k}=d_k/D_n$ for $0\leqslant k\leqslant n$ is a weighted mean matrix when $d_0=1$ and otherwise differs in only a minor way from a weighted mean matrix. Conversely, every weighted mean matrix with positive weights may be regarded, in view of (5), (6), (7), and (8), as a generalized Hausdorff matrix with $\lambda_0=0$. The matrix M_d is regular if and only if $D_n\to\infty$. Note the following equivalences:

$$D_n\to\infty \text{ is equivalent to } \sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty;$$

$$d_n/D_n\to 0 \text{ is equivalent to } \lambda_n\to\infty;$$

$$d_n/D_n\searrow \text{ is equivalent to } \lambda_n\nearrow.$$

Regularity

In this section conditions are established for the regularity of generalized Hausdorff matrices. The following lemma is required; it concerns matrices $\{\lambda_{n,k}\}$ given by (2) with the function f satisfying, for some real number c, a condition of the form

$$(-1)^r f^{(r)}(x)\geqslant 0 \qquad \text{for} \quad r=0,1,2,\ldots \text{ and } x>c; \tag{9}$$

and the region Ω, in which f is holomorphic, satisfying the condition

$$\Omega\supset(c,\infty). \tag{10}$$

Lemma 1. (i) *If* (9) *and* (10) *hold with* $c=0$, *then* $l_k=\lim_{n\to\infty}\lambda_{n,k}$ *exists for* $k=0,1,2,\ldots$. *If, in addition,*

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty, \tag{11}$$

then $l_k=0$ *for* $k=1,2,3,\ldots$, *and* $l_0=0$ *if* $\lambda_0>0$.

(ii) *If* $\lambda_0=0$, (9) *and* (10) *hold with* $c=-\varepsilon$ *for some* $\varepsilon>0$, *and* (11) *holds, then* $l_0=0$.

Proof. If $a\leqslant\lambda_\nu\leqslant b$ for $k\leqslant\nu\leqslant n$, then

$$-\frac{1}{2\pi i}\int_{\Gamma_n}\frac{f(z)\,dz}{(\lambda_k-z)\cdots(\lambda_n-z)}=\frac{(-1)^{n-k}}{(n-k)!}f^{(n-k)}(\xi) \tag{12}$$

for some $\xi \in [a, b]$. (See Lorentz [5].) Further, the recursion formula

$$\lambda_{n,k} - \lambda_{n+1,k} = (\lambda_{k+1}\lambda_{n+1,k+1} - \lambda_k \lambda_{n+1,k})/\lambda_{n+1} \quad \text{for } 0 \leqslant k \leqslant n$$

is an immediate consequence of (2). Letting $\Lambda_{n,k} = \sum_{v=0}^{k} \lambda_{n,v}$ for $0 \leqslant k \leqslant n$, it follows, as in Hausdorff [3], that

$$\Lambda_{n,k} - \Lambda_{n+1,k} = (\lambda_{k+1}\lambda_{n+1,k+1} - \lambda_0 \lambda_{n+1,0})/\lambda_{n+1}. \tag{13}$$

Suppose now that $\lambda_0 = 0$, then, by (12), $\lambda_{n+1,k+1} \geqslant 0$ so that, by (13), $\Lambda_{n,k} \geqslant \Lambda_{n+1,k} \geqslant 0$. Hence, $L_k = \lim_{n\to\infty} \Lambda_{n,k}$ exists and so does $l_k = L_k - L_{k-1} = \lim_{n\to\infty} \lambda_{n,k}$ (with $L_{-1} = 0$). Equation (13) also shows that, for $k = 0, 1, 2,\ldots$, the series $\sum_{n=0}^{\infty} \lambda_{n+1,k+1}/\lambda_{n+1}$ is convergent; consequently, by (11), $l_k = 0$ for $k = 1, 2,\ldots$.

Next, suppose that $\lambda_0 > 0$. Define $\bar{\lambda}_0 = 0$ and $\bar{\lambda}_n = \lambda_{n-1}$ for $n = 1, 2, 3,\ldots$, and define $\bar{\lambda}_{n,k}$ in the same way as $\lambda_{n,k}$ but with $\bar{\lambda}_n$ replacing λ_n. Then $\lambda_{n,k} = \bar{\lambda}_{n+1,k+1}$ for $0 \leqslant k \leqslant n$, and hence $l_k = \lim_{n\to\infty} \lambda_{n,k} = \bar{l}_{k+1} = \lim_{n\to\infty} \bar{\lambda}_{n+1,k+1} = 0$ for $k = 0, 1, 2,\ldots$. This establishes (i).

Suppose now the hypotheses of (ii) hold. Then, for sufficiently small positive η,

$$\begin{aligned} \lambda_{n,0} &= -\lambda_1\lambda_2\cdots\lambda_n \frac{1}{2\pi i}\int_{\Gamma_n} \frac{f(z)\,dz}{-z(\lambda_1 - z)\cdots(\lambda_n - z)} \\ &= -(\lambda_1 + \eta)\cdots(\lambda_n + \eta)\frac{\gamma_n}{2\pi i} \\ &\quad \times \int_{\Gamma_n} \frac{f(z-\eta)\,dz}{(\eta - z)(\lambda_1 + \eta - z)\cdots(\lambda_n + \eta - z)} \end{aligned}$$

where $0 \leqslant \gamma_n = \lambda_1\lambda_2\cdots\lambda_n/(\lambda_1 + \eta)\cdots(\lambda_n + \eta) \leqslant 1$. Since $\sum_{n=1}^{\infty} 1/(\lambda_n + \eta) = \infty$ when (11) holds, it follows from the earlier part of the proof that $l_0 = 0$. This completes the proof.

It follows from Lemma 1, with $f(z) = t^z$, and from (4) that, for $0 \leqslant t \leqslant 1$,

$$\begin{aligned} \lim_{n\to\infty} \lambda_{n,k}(t) &= 1 \qquad \text{if } t = 0,\ k = 0, \text{ and } \lambda_0 = 0, \\ &= 0 \qquad \text{otherwise.} \end{aligned} \tag{14}$$

Next, for $\lambda_0 = 0$, one has [5, p. 46]

$$\frac{1}{z} = -\sum_{k=0}^{n} \frac{\lambda_{k+1}\cdots\lambda_n}{(\lambda_k - z)\cdots(\lambda_n - z)},$$

and hence if $0 \in \Omega$ (so that f is holomorphic at 0), then

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{f(z)}{z} \, dz = f(0) = \sum_{k=0}^{n} \lambda_{n,k}.$$

For $\lambda_0 > 0$, put $\tilde{\lambda}_0 = 0$, $\tilde{\lambda}_n = \lambda_{n-1}$ for $n \geqslant 1$, to get

$$\sum_{k=0}^{n} \lambda_{n,k} = \sum_{k=0}^{n+1} \tilde{\lambda}_{n+1,k} - \tilde{\lambda}_{n+1,0} \to f(0)$$

provided that $\tilde{\lambda}_{n+1,0} \to 0$. In particular, with $f(z) = t^z$, this, together with (12), yields

$$0 \leqslant \lambda_{n,j}(t) \leqslant \sum_{k=0}^{n} \lambda_{n,k}(t) \leqslant 1 \tag{15}$$

and, in view of (4),

$$\begin{aligned} \lim_{n \to \infty} \sum_{k=0}^{n} \lambda_{n,k}(t) &= 0 \qquad \text{if } t = 0 \text{ and } \lambda_0 > 0, \\ &= 1 \qquad \text{otherwise.} \end{aligned} \tag{16}$$

Borwein and Jakimovski show in [1] that if (11) holds and $\lambda_n \to \infty$, then for the matrix given by (2) to be regular, it is necessary that $f(z) = \int_0^1 t^z \, d\alpha(t)$ for some $\alpha \in BV$. There is thus no real loss in so restricting f in the following theorem.

THEOREM 1. *Suppose that* (11) *holds and* $f(z) = \int_0^1 t^z \, d\alpha(t)$ *for some* $\alpha \in BV$ *with*

$$\alpha(1) - \alpha(0) = 1 \tag{17}$$

and

$$\alpha(0+) = \alpha(0). \tag{18}$$

Then the matrix $\{\lambda_{n,k}\}$ *defined by* (2) *is regular.*

Proof. By Lebesgue's theorem on bounded convergence, it follows from (14) and (18) that, for $k = 0, 1, 2, \ldots$,

$$\lambda_{n,k} = \int_0^1 \lambda_{n,k}(t) \, d\alpha(t) \to 0 \qquad \text{as } n \to \infty;$$

and from (16) and (18) that

$$\sum_{k=0}^{n} \lambda_{n,k} = \int_0^1 \left(\sum_{k=0}^{n} \lambda_{n,k}(t) \right) d\alpha(t) \to \int_0^1 d\alpha(t) \qquad \text{as } n \to \infty.$$

Also, from (15),

$$\sum_{k=0}^{n} |\lambda_{n,k}| \leqslant \int_0^1 |d\alpha(t)|.$$

In view of (17), the matrix is regular.

Generalized Hölder and Cesàro Matrices

The next lemma concerns products of certain matrices.

LEMMA 2. *Suppose that g and h are holomorphic in Ω and are defined at λ_0 even when $\lambda_0 \notin \Omega$. Let A, B, and C be the Hausdorff matrices given by* (2) *with f replaced by g, h, and gh, respectively. Then $C = AB$.*

Proof. It is sufficient to establish the result for $\lambda_0 = 0$ since the general result then follows in the usual manner by defining $\tilde{\lambda}_0 = 0$ and $\tilde{\lambda}_n = \lambda_{n-1}$ for $n \geqslant 1$. For $m = 0, 1, 2,\ldots$, let A_m, B_m, and C_m be the principal $m \times m$ minors of the matrices A, B, and C, respectively. It is now sufficient to show that $C_m = A_m B_m$. Suppose first that λ_0, $\lambda_1,\ldots$, λ_m are distinct. Then, as in Hausdorff [3], there is a matrix ρ such that $A_m = \rho^{-1}\alpha\rho$, $B_m = \rho^{-1}\beta\rho$, and $C_m = \rho^{-1}\alpha\beta\rho$, where α and β are the diagonal matrices with $g(k)$ and $h(k)$, respectively, in the kth position along the diagonal. Thus $C_m = A_m B_m$, and a continuity argument shows that this equation remains valid if certain of the λ_ν's are allowed to coalesce. This completes the proof.

For κ real, the Hölder matrix H_κ is the generalized Hausdorff matrix obtained from (2) by taking

$$f(z) = (z+1)^{-\kappa}.$$

For $\kappa > -1$, the Cesàro matrix C_κ is the generalized Hausdorff matrix obtained from (2) by taking

$$f(z) = \frac{\Gamma(\kappa+1)\,\Gamma(z+1)}{\Gamma(z+\kappa+1)}.$$

Hausdorff, in [3], showed that if $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$, $\sum_{n=1}^{\infty} (1/\lambda_n) = \infty$, and $\kappa > -1$, then H_κ and C_κ are quivalent: i.e., $s_n \to s(H_\kappa)$ if and only if $s_n \to s(C_\kappa)$. It is now easy to extend the result as follows.

THEOREM 2. *Suppose* $\lambda_0 \geqslant 0$, $\lambda_n > 0$ *for* $n \geqslant 1$, $\sum_{n=1}^{\infty} (1/\lambda_n) = \infty$, *and* $\kappa > -1$. *Then* H_κ *and* C_κ *are equivalent.*

Proof. Let

$$g(z) = \frac{\Gamma(\kappa+1)\,\Gamma(z+1)}{\Gamma(\kappa+z+1)}\,(z+1)^{\kappa}.$$

It follows from results of Rogosinski [6, pp. 188ff., 167] that $g(z) = \int_0^1 t^z\, d\alpha_1(t)$ and $1/g(z) = \int_0^1 t^z\, d\alpha_2(t)$ where $\alpha_i \in BV$, $\alpha_i(0+) = \alpha_i(0)$, and $\alpha_i(1) - \alpha_i(0) = 1$ for $i = 1, 2$. The desired conclusion now follows from Theorem 1 and Lemma 2.

REFERENCES

1. D. BORWEIN AND A. JAKIMOVSKI, *Generalization of the Hausdorff moment problem*, *Canad. J. Math.* **33** (1981), 946–960.
2. F. HAUSDORFF, *Summationsmethoden und Momentfolgen*, I, *Math. Z.* **9** (1921), 74–109.
3. F. HAUSDORFF, *Summationsmethoden und Momentfolgen*, II, *Math. Z.* **9** (1921), 280–299.
4. G. H. HARDY, "*Divergent Series*," Oxford Univ. Press, London/New York, 1949.
5. G. G. LORENTZ, "*Bernstein Polynomials*," Univ. of Toronto Press, Toronto, 1953.
6. W. W. ROGOSINSKI, *On Hausdorff methods of summability*, *Proc. Cambridge Philos. Soc.* **38** (1942), 166–192.